String tension and removal of lattice coarsening effects in Monte Carlo Renormalization Group

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Abstract

We study the computation of the static quark potential under decimations in the Monte Carlo Renormalization Group (MCRG). Employing a multi-representation plaquette action, we find that fine-tuning the decimation prescription so that the MCRG equilibrium self-consistency condition is satisfied produces dramatic improvement at large distances. In particular, lattice coarsening (change of effective lattice spacing on action-generated lattices after decimation) is nearly eliminated. Failure to correctly tune the decimation, on the other hand, produces large coarsening effects, of order 50% or more, consistent with those seen in previous studies. We also study rotational invariance restoration at short distances, where no particular improvement is seen for this action.

1   Introduction

The construction and study of improved lattice actions has received a good deal of attention over the years. In the Wilsonian renormalization group (RG), starting from a suitable cut-off action, a blocking transformation results into an improved action in the sense of being closer to the RG renormalized trajectory. In the Monte Carlo Renormalization Group (MCRG) approach some block-spinning transformation is applied to a configuration ensemble obtained by Monte Carlo simulations. The resulting blocked ensemble is then assumed to be Boltzmann-distributed according to some effective action defined on the decimated lattice. One, however, does not know at the outset what this improved action is. The standard procedure has been to adopt a model for it, and then proceed to measure its couplings on the decimated configurations by the demon [1] or some other method.

Any effective action model is necessarily restricted to some subspace of interactions. This implies that one is always faced with the problem of truncation

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effects, i.e. the issue of whether the space of interactions retained in the model is sufficient to adequately describe the decimated ensemble at least over some scale regime. As demonstrated in our recent work [2, 3, 4], however, there is another issue that must always be separately considered. It pertains to the self-consistent application of the MCRG method itself: the decimated ensemble must belong to the equilibrium ensemble of the action model at the couplings obtained by measurements on the decimated ensemble. We refer to this require-ment as the MCRG equilibrium self-consistency condition. Correct application of the MCRG method entails checking whether this condition is satisfied. Given some choice of blocking prescription and effective action model, one will in fact find that, in general, the condition is *not* satisfied.

One obvious reason for this may be that the chosen model of the blocked effective action simply cannot adequately approximate the blocked ensemble over any scale regime. In such a case the truncation errors completely dominate. There is then nothing to do but include an appropriate wider class of interactions in the effective action model.

Another, more subtle cause for the inconsistency, however, may be in play [3]. The 'true' blocked action depends on the precise choice of decimation pre-scription.[1] It may then be that the assumed effective action model, though in principle a quite adequate approximation over some scale regime, has not yet been appropriately matched to the chosen decimation procedure in the fol-lowing sense. Commonly employed lattice decimation prescriptions typically involve adjustable parameters, such as the weights of different kinds of staples formed out of the bond variables on the undecimated lattice. In general, the above equilibrium consistency condition can be satisfied, *if at all*, only for par-ticular (range of) values of these parameters [3]. In other words, fine-tuning of the decimation prescription is needed to match to some appropriate effective action over some scale regime.

This state of affairs, which is generic in the application of MCRG, was ex-tensively demonstrated in [2, 3, 4] in decimation studies in $SU(2)$ lattice gauge theory (LGT). In these studies, two alternative standard decimation proce-dures were employed: Swendsen decimation [5], and "double smeared blocking" (DSB) [6] decimation. Both decimation prescriptions involve a free parameter c, which is the weight of staples relative to straight paths in the construction of the decimated lattice bond variables out of the original lattice bond variables. Two different effective action models were explored: the multiple-representation single plaquette action (eq. (1) below)), and the fundamental representation plaquette-plus-rectangle (1×2 planar loop) action.

In the case of the multi-representation plaquette action it was found that for DSB decimation the parameter c can be sharply fine-tuned so that the MCRG equilibrium condition is satisfied. The same is true for Swendsen decimations, though there the fine-tuning is somewhat less sharp. Overall, DSB decimation

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[1]Indeed, within the general Wilsonian renormalization group framework, the precise form of the action resulting from a block-spinning transformation cannot be divorced from the choice of blocking transformation: different choices for the definition of blocked field variables, in terms of the original field variables, will, in general, result in different blocked effective actions.

turns out to be better suited for this action. By comparing the values of $N \times N$ loop observables measured on the decimated ensemble, denoted $W_{N \times N}^{dec}$, to those measured on the effective-action-generated ensemble, denoted $W_{N \times N}^{gen}$, substantial improvement with increasing length scale was found, as expected, at the optimal (fine-tuned) c value [3]. 'Improvement' here means that the difference between $W_{N \times N}^{dec}$ and $W_{N \times N}^{gen}$ is reduced (ideally goes to zero), which implies that the blocked ensemble is adequately represented by the effective action model.

In the case of the plaquette-plus-rectangle fundamental representation action no c value was found such that DSB-decimated configurations are nearly equilibrium configurations of the action. Such a value, however, could be found for Swendsen decimations.[2] Comparison of the (weighted) difference of $W_{N \times N}^{dec}$ and $W_{N \times N}^{gen}$ shows that the plaquette-plus-rectangle works well (nearly vanishing difference) at short distances, but gives consistent growth of the difference with increasing N towards intermediate distances; it thus appears to fail as an effective action at intermediate to long distances. The multi-representation plaquette action at the optimal c, on the other hand, shows improvement with increasing length scales, indicating improved efficacy as a longer scale effective action.

In [2], [3] observables up to size $N = 8$ (in undecimated lattice units) were investigated. In this paper we consider a wider range of scales in measurements of the Polyakov line correlator in $SU(2)$ LGT. We obtain the static quark-antiquark potential which allows us to probe different scales and extract the string tension. Comparison of string tensions on the blocked and effective-action-generated ensembles allows us then to test the extent to which the assumed effective action represents the blocked ensemble in terms of a long-distance physical quantity. Dramatic improvement is found using the multi-representation plaquette action after fine-tuning to the c value that satisfies the equilibrium self-consistency requirement. For any c values away from this optimal value, on the other hand, large deviations, i.e. sizable effective changes of scale (coarsening), are found. The size of the coarsening effects found away from the optimal c value is consistent with that reported in previous studies [7].

We also test for rotational invariance improvement at short distances. No actual improvement is found with the multi-representation plaquette action. This action then appears to provide an effective long distance description under DSB decimation at the expense of some distortion at short scales. Our findings are further discussed in section 4.

2 String tension and coarsening under decimation

We start with the $SU(2)$ fundamental representation Wilson action at $\beta = 2.5$ and perform DSB decimations. We take the multi-representation plaquette

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[2]Overall, however, this action exhibits erratic behavior under variation of c, in contrast to the smooth, monotonic behavior of the multi-representation plaquette action under such variation. This probably indicates instability under addition of other terms.

action

$$S = \sum_{j=1/2}^{j_N} \beta_j [1 - \frac{1}{d_j}\chi_j(U_p)] \, , \qquad (1)$$

as our effective action model, which, as reviewed in the previous section, appears better suited for representing long distance properties of the blocked configurations. In (1) χ_j denotes the j-representation character, and β_j the corresponding coupling. U_p stands for the product of bond variables around a plaquette p. For simulating the multi-representation action (1) we use the procedure in [8]. To measure couplings we use the microcanonical method [1]. For microcanonical updating and demon measurements we use the algorithm in [9]. We refer to [3], [2] for details of our computational procedures and numerical simulations. The couplings after one DSB decimation with staple weight parameter c and scale factor $b = 2$ were computed for various values of c in [3] (Table V). They are reproduced here for some c values in Table 1.

c	$\beta_{1/2}, \beta_1, \beta_{3/2}, \beta_2, \ldots$
0.060	2.4660(7), -0.3635(11), 0.1242(17), -0.0475(21), 0.0195(25), -0.0070(24)
0.065	2.5023(7), -0.3098(12), 0.1057(16), -0.0397(16), 0.0145(14), -0.0029(15)
0.067	2.5125(7), -0.2832(16), 0.0964(25), -0.0367(29), 0.0139(29)
0.068	2.5183(9), -0.2701(13), 0.0916(16), -0.0351(17),0.0142(17),-0.0053(20)
0.077	2.5463(11), -0.1167(17), 0.0320(23), -0.0055(28)

Table 1: The couplings of the effective action (1) after DSB decimation starting from the Wilson action at $\beta = 2.5$. c is the staple weight.

We compute the static quark potential from the Polyakov line correlator. The string tension σ is then extracted, as usual, by fitting the potential $V(r)$ to the expression

$$V(r) = m - \frac{\mu}{r} + \sigma r \, , \qquad (2)$$

where r is the distance in lattice units.

We start on $32^3 \times 12$ lattice with the Wilson action at $\beta = 2.5$, and obtain the potential $V_0(r)$ and string tension σ_0. We next perform a DSB decimation with scale factor $b = 2$ and several choices of the c parameter. For each such c value we measure the potential V_{dec} and the string tension σ_{dec} on the decimated ensemble on the resulting $16^3 \times 6$ lattice. We then generate configurations for the effective action (1) at the couplings obtained after the decimation (Table 1). The potential V_{gen} and the string tension σ_{gen} are then obtained from this effective-action-generated ensemble. For decimated potential measurements we use 30 replicas of runs, each typical run consisting of up to 10000 sweeps

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with measurement at every 3rd sweep, and 1 heatbath and 2 overrelaxations per sweep. For the effective action potential measurements shorter runs, each typically of 100 sweeps, are used with measurements every 3rd sweep. For measurements on the original and on the generated ensembles, for which the action is known, we use the Lüscher-Weisz technique; whereas, on the decimated ensemble, for which the action is not known, we use 'naive' straightforward correlation measurements, which explains the need for rather longer runs. Also for this reason, we chose the original lattice at relatively high temperature so that the correlator decays slower, allowing us to use the 'naive' methods on the decimated ensemble. The values of the string tensions σ_0, σ_{dec} and σ_{gen} obtained are listed in Table 2. One sees that the DSB decimated configurations produce

c	$\sigma_0 = 0.0313(2)$	
	σ_{dec}	σ_{gen}
0.060	0.0271(37)	0.0594(12)
0.065	0.0284(30)	0.0385(14)
0.067	0.0291(49)	0.0346(8)
0.068	0.0295(12)	0.0292(9)
0.077	0.0285(24)	0.0091(6)*

Table 2: String tensions in original (undecimated) lattice units. σ_0 on $32^3 \times 12$, σ_{dec}, σ_{gen} on $16^3 \times 6$, except starred entry which is on $16^3 \times 8$.

correct values of the string tension over the range of c values shown. (This is also the case for Swendsen decimations, cf. Table I in [3].) The string tension σ_{gen} extracted from the effective-action-generated ensemble, however, shows marked dependence on the decimation parameter c. In fact, for $c = 0.077$ the time-like lattice extension had to be increased to $N_t = 8$ to keep the generated ensemble in the confined phase.

In Fig. 1 we plot the static quark potential $V_0(r)$ obtained on the original (undecimated) $32^3 \times 16$ lattice. To satisfy the MCRG equilibrium self-consistency condition the c parameter has to be fine-tuned in the near vicinity of the value $c = 0.067$ (see [3]). The potentials V_{dec} and V_{gen} obtained after DSB decimation for this optimal value of c are plotted for comparison also in Fig. 1. The potential V_{dec} obtained from the decimated ensemble differs from V_0 in the constant term m representing mass renormalization per unit length of the Polyakov lines (external sources). Discrepancies in the Coulomb μ coefficient value (cf. [3]) result in some additional distortion, which, however, becomes unimportant with increasing r. This is as expected; numerical decimation procedures typically introduce some short distance distortion. After a shift by a constant, represented by the relatively shifted l.h.s. and r.h.s. vertical axes in Fig. 1, V_{dec} falls for the most part nearly on top of V_0. This reflects the fact that the string tension is well reproduced in V_{dec}.

The important feature of Fig. 1, however, is that V_{gen}, computed at the fined-tuned value $c = 0.067$, closely tracks V_{dec} and V_0 over a wide distance range. Note that this occurs without the need for any constant shift between
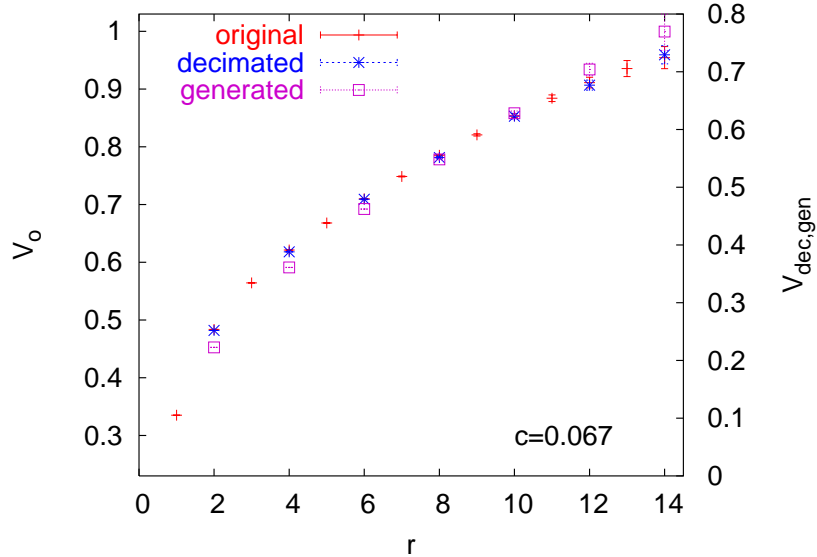
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Figure 1: Static quark potential V_0 computed on original (undecimated), V_{dec} on DSB decimated ($c = 0.067$), and V_{gen} on effective-action-generated lattices.

V_{dem} and V_{gen}, which means that the generated ensemble produces nearly the same mass renormalization effect m as the decimated effect. The near agreement, beyond short distance effects, of V_{gen} with V_{dec} and V_0 means then that it also reproduces the string tension well, as evident from Table 2. The Gaussian difference test for the string tension values on decimated and generated lattices gives $Q = 0.27$ (probability that the discrepancy is due to chance), an indication of good agreement of data. To quantify this agreement in physical terms, we consider the actual change in scale that resulted from the blocking operations beyond the expected scale change by the blocking factor $b = 2$. To this end we examine the ratios among the lattice spacings in the original, decimated and action-generated lattices:

$$\frac{a_{dec}}{2a} = \sqrt{\frac{\sigma_{dec}}{\sigma_0}}, \qquad \frac{a_{gen}}{a_{dec}} = \sqrt{\frac{\sigma_{gen}}{\sigma_{dec}}}, \qquad \frac{a_{gen}}{2a} = \sqrt{\frac{\sigma_{gen}}{\sigma_0}}, \tag{3}$$

where a denotes the lattice spacing on the original (undecimated) lattice. If the decimation procedure and the identification of the effective action were exact, these ratios would all be equal to one. Deviations from unity, which are commonly referred to as 'coarsening' errors, amount to a change of lattice scale in addition to that by the blocking factor b (equal to 2 here). At our fine-tuned value $c = 0.067$, one has $a_{gen}/a_{dec} = 1.09(9)$, $a_{gen}/2a = 1.051(13)$, and $a_{dec}/2a = 0.96(8)$. (Here errors are computed using standard error propagation.)

Remarkably, then, once the decimation procedure is fine-tuned to satisfy

the MCRG equilibrium self-consistency condition with the multi-representation plaquette action, there is virtually no coarsening effect on the blocked lattice.

This is in sharp contrast to what happens at other values of c, i.e. when the condition is not satisfied. The values for the ratios (3) for different values of c are displayed in Table 3. Even relatively small deviations outside a small window

c	$a_{dec}/2a$	a_{gen}/a_{dec}	$a_{gen}/2a$
0.060	0.93(6)	1.48(10)	1.378(15)
0.065	0.95(5)	1.16(7)	1.109(20)
0.067	0.96(8)	1.09(9)	1.051(13)
0.068	0.97(2)	0.99(3)	0.966(15)
0.077	0.95(4)	0.57(3)	0.539(18)

Table 3: Lattice spacings ratios among original, decimated and effective-action-generated lattices.

inside the interval $0.0065 < c \lesssim 0.0068$ result in sizable coarsening effects. Thus, at $c = 0.060$ one has $a_{gen}/a_{dec} = 1.481$, i.e. a coarsening effect of about 50% of the action-generated lattice compared to the decimated lattice. For illustrative purposes, we also plot in Fig. 2 the static quark potentials V_0, V_{dec} and V_{gen} at $c = 0.060$. The contrast with Fig. 1 is manifest. Such change-of-scale effects of order $40 - 50\%$, and corresponding deviations in the blocked potentials as those seen in Fig. 2 are typical of previous MCRG studies fixing the decimation parameters on an ad-hoc basis. For the 'classical' value $c = 0.077$, which has been used before, for example, the coarsening effect is $\sim 75\%$!

3 Rotational invariance restoration

Our decimations have a blocking factor equal to 2. The Wilson action on a lattice of spacing $2a$ would have a string tension $\sigma = (2a/a)^2\sigma_0$, i.e. four times the original string tension. Therefore, we expect $\sigma \sim 0.1252$. This is very close to the string tension for the Wilson action at $\beta = 2.31$, which is $\sigma = 0.1230(14)$. We compare departures from rotational symmetry on this Wilson action lattice and on the effective-action-generated lattice. The time-like Polyakov lines of the correlator intersect a 3-dimensional space-like lattice slice at two points. We consider on-axis separation between these two points in the direction 100, and off-axis separations in the directions 110 and 111 (in terms of unit vectors in the space-like slice). The results are presented in Fig. 3.

We quantify the amount of rotational invariance violation by [10]:

$$\delta_V^2 = \sum_{off} \left(\frac{V_{off}(r) - V_{on}(r)}{V_{off}(r)\delta V_{off}(r)} \right)^2 / \sum_{off} \frac{1}{\delta V(r)^2}. \tag{4}$$

We get[3] $\delta_V = 0.045$ for Wilson, and $\delta_V = 0.047$ for the effective action.

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[3]The first four off-axis points, counting from shortest distance, in Fig. 3 were included in the sums in (4).
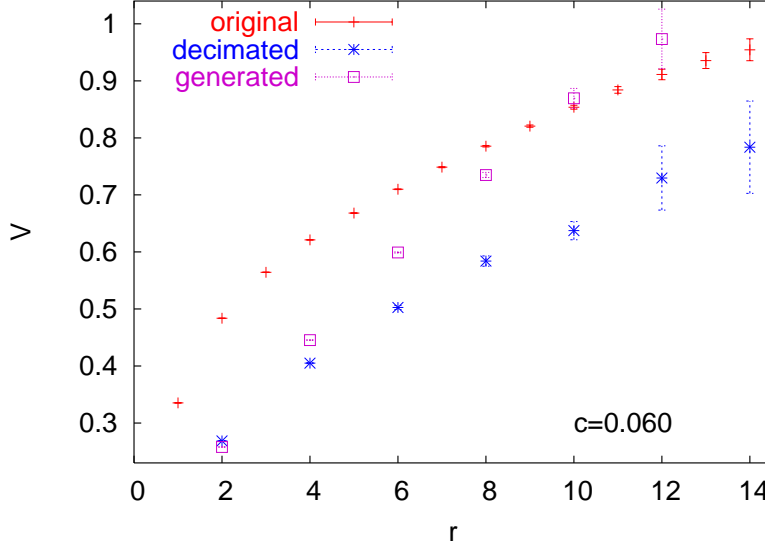
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Figure 2: Static quark potential V_0 computed on original (undecimated), V_{dec} on DSB decimated ($c = 0.060$) and V_{gen} on effective-action-generated lattices.

They are comparable - there is no improvement at short distance.

4   Summary and Conclusions

In the MCRG method one has to assume a model for the effective action on the blocked lattice. Consistent application of the MCRG method then involves checking whether the blocked ensemble belongs to the equilibrium ensemble generated from the effective action model. In general this test will fail. This may indicate that the action model has been restricted to a set of interactions that do not adequately approximate the true action representing the blocked ensemble, i.e. truncation effects are paramount. But, as was pointed out in [2], [3], another possible cause for this failure may be that the decimation procedure and the assumed effective action have not been properly matched. This means that the action may in fact be an adequate approximation in some scale regime, but any adjustable parameters that enter in the specification of the decimation prescription have not been properly tuned so that the equilibrium self-consistency condition is satisfied.

In this paper we explored this effect by computing the static quark potential and extracting the string tension on the original, the decimated and the effective-action-generated ensembles. Discrepancies between the string tensions from the different ensembles amount to a change in scale in addition to the rescaling from a lattice of spacing a to a decimated lattice of spacing $2a$ (for a blocking by a
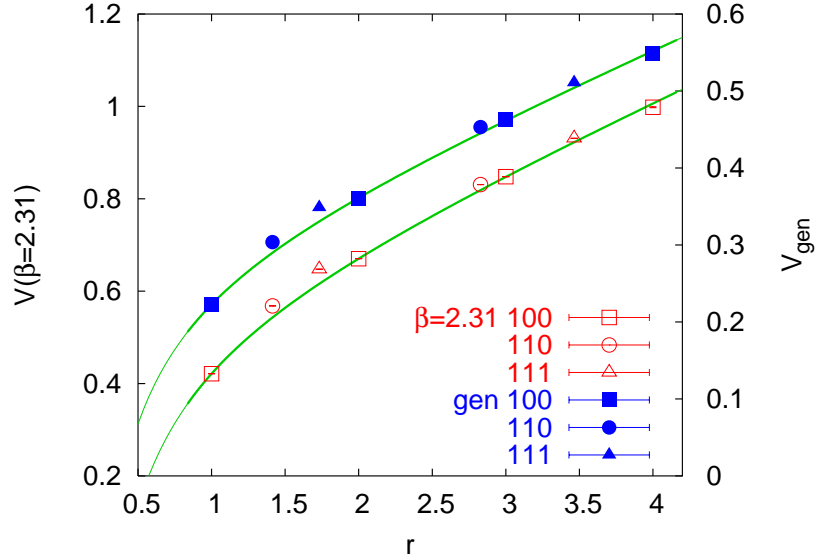
Figure 3: Static quark-antiquark potential on the effective-action-generated and on $\beta = 2.31$ Wilson lattices; on and off axis separations.

factor of 2). Such discrepancies are referred to as lattice coarsening errors.

Employing DSB decimations with the multi-representation plaquette action model we found that fine-tuning the DSB parameter c so that the MCRG equilibrium self-consistency condition is fulfilled has a rather dramatic effect: lattice coarsening is largely eliminated. On the other hand, even small departures from this value of c result into sizable deviations between the potentials on the different ensembles and correspondingly sizable lattice coarsening effects. Coarsenings of 50% or more are typical. One can conclude that for the multi-representation plaquette action such coarsening arises almost entirely from non-equilibrium rather than actual truncation effects.

Since any effective action model will not be quite exact, some truncation errors, of size generally varying with distance scale, are always present. We tested the multi-representation plaquette action for rotational invariance at short distances. No improvement in rotational invariance restoration at short distances was observed compared to the Wilson action.

In conclusion, the $SU(2)$ multi-representation plaquette action provides a good long-distance representation of the ensemble obtained by DSB decimation, at least as far as observables like the static quark potential are concerned. For improvement at short distances it has to be augmented, presumably by including loops larger than the plaquette. It would clearly be very worthwhile to carry out such a program in the case of $SU(3)$.

More generally, a main conclusion of this study is that, no matter what its form is, no correct assessment of an adopted effective action can be obtained

without properly tuning the family of decimations being employed to it. The purpose of this tuning must be to satisfy, to the extent possible, the MCRG equilibrium self-consistency condition. Failure to do this can completely obscure the actual efficacy of the action to represent the blocked ensemble and the origin of the errors involved.

Acknowledgments

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